Some Math Tools for Signals and Systems

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The following mathematical concepts are important to a course on signals and systems as well as engineering in general. The student should know these concepts well. For rigorous mathematical proofs, which are omitted and not essential for a rudimentary introductory course on signals and systems, see any of several good texts on mathematical analysis such as Rudin [1].

1 Variable notation

It is good practice to choose notation for variables that is consistent with the pseudo-standard that has developed in mathematics. Table 1 gives conventions that are generally true. However, one should always make clear what type of numbers, functions or other entities the variables chosen represent.

2 Completing the square

Completing the square is an algebraic technique for writing a second order polynomial variable or expression as a squared term minus the quantity necessary to *complete the square*. Deriving the quadratic formula is an excellent example of completing the square. A quadratic *equation* is one which can be put in the form

$$au^2 + bu + c = 0, \tag{1}$$

where a, b and c are *real* constants and $a \neq 0$. The symbol $\mathbb{R}$ will be used to represent the set of real numbers. Here u need not be a variable, but could be a mathematical expression. Although not necessary, dividing (1) by a makes visualizing the procedure a bit easier.

$$\begin{aligned} u^2 + \frac{b}{a}u + \frac{c}{a} &= u^2 + \frac{b}{a}u + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} \\ &= \left(u + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} \\ &= 0. \end{aligned}$$

Having completed the square, it follows that

$$\left(u + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2},$$

or

$$u + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}},$$

or

$$u = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2|a|}. \tag{2}$$

Given that a is a real number, the absolute value of (2) is extraneous since there is a $\pm$ before the radical. Hence, the quadratic *formula* becomes

$$\boxed{u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.} \tag{3}$$

Variable	Typical symbols	Typeset format	Chalkboard format
Integer	i, j, k, l, m, n sometimes p, q	Lower case	
Real or complex	$a, b, c, d, s, t, w, x, y$	Lower case	
Complex	z	Lower case	
Function	f, g, h, y, F	Lower case, sometimes upper case	
Summation	S	Upper case, nonbold	
Vector	$\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}, \ldots$	Boldface, lower case, roman	Lower case, underbar or overbar, i.e., $\overline{\mathrm{x}}$
Matrix	$\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{X}, \mathbf{Y}, \ldots$	Boldface, upper case, roman	Upper case
Operator limits	a, b, c, d	Lower case	
Parameters	α, β, γ	Lower case Greek	

Table 1: Conventions for mathematical notation. These are general rules that are some times bent.

3 Complex numbers

The derivation of the completing the square example did not specify u, although a, b and c where taken as real. If u is considered a variable it is worthwhile to consider the properties of a solution to the quadratic equation. The quadratic formula of (3) yields characteristically different solutions based upon the magnitude of $b^2 - 4ac$ inside the radical. If $b^2 - 4ac$ is positive then there exist two distinct real roots of the quadratic. If $b^2 - 4ac$ is zero then there is a repeated real root. Lastly, if $b^2 - 4ac$ is negative there is no real number solution to the quadratic. The complex number has been devised to provide a solution in the latter case.

A **complex number** is an ordered pair of real numbers with addition and multiplication defined as

$$\text{A:} \quad (a, b) + (c, d) = (a + c, b + d) \tag{4}$$

$$\text{M:} \quad (a, b) \cdot (c, d) = (ac - bd, ad + bc). \tag{5}$$

Here (a, b) and (c, d) are ordered pairs. An *ordered pair* means that $(a, b) \neq (b, a)$ in general. It is somewhat cumbersome to work with the notation of (4) and (5) so it is usually the convention to make the following definitions:

$$\begin{aligned} 0 &= (0, 0), \quad \text{(Additive identity)} \\ 1 &= (1, 0), \quad \text{(Multiplicative identity)} \\ j &= (0, 1), \end{aligned}$$

and it then follows that $j^2 = -1$ and $(a, b) = a + jb$. With this convention we have the familiar rules

$$\begin{aligned} \text{A:} \quad & (a + jb) + (c + jd) = (a + c) + j(b + d) \\ \text{M:} \quad & (a + jb) \cdot (c + jd) = ac + jad + jbc + j^2 bd \\ & \qquad = (ac - bd) + j(ad + bc). \end{aligned}$$

The complex numbers are complete in an algebraic sense as summarized by the following theorem.

FUNDAMENTAL THEOREM OF ALGEBRA: Counting multiple roots, every polynomial of degree n having complex coefficients has exactly n roots in the set of complex numbers.

Hence, the roots of (1) always exist in the set of complex numbers. Note that the quadratic formula as derived in (3) uses real coefficients and one must be careful not to use this formula to compute roots for a complex coefficient equation.

The real and imaginary parts of a complex number $z = a + jb$ are defined as a and b, respectively, and denoted as

$$\text{Re}\{z\} = a, \quad \text{Im}\{z\} = b.$$

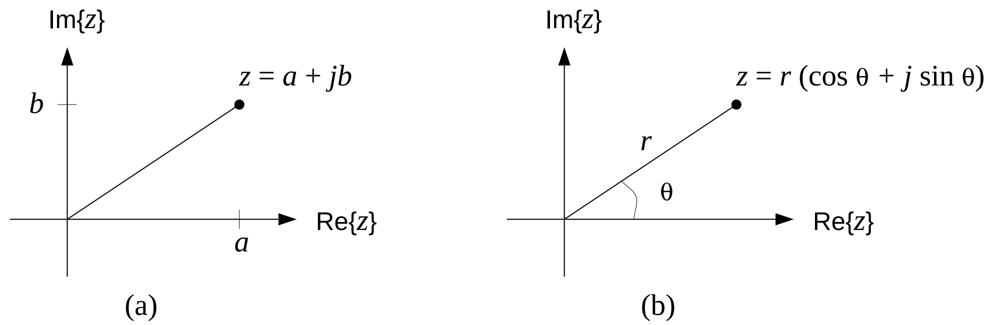

Figure 1: The complex plane showing the real and imaginary parts of a complex number. This is often called the z-plane. (a) Cartesian form and (b) polar form representation.

Note that the imaginary part of a complex number is b alone, while the quantity jb is called an imaginary number. Conceptualizing a *complex plane* where the real and imaginary parts play the role of a vector dimension is beneficial. Figure 1(a) shows this plane, also called the z-plane. The *complex conjugate* of a complex number z is defined as

$$\overline{z} = \text{Re}\{z\} - j\ \text{Im}\{z\} = a - jb.$$

One thing that is lost upon extending the real numbers to complex numbers is the ordered property. That is, the statements $z < y$ or $z > y$ are no longer proper when z and y are complex numbers. Hence it is necessary to impose some type of order onto complex numbers. Since $z\overline{z} = a^2 + b^2$ is a nonnegative real number, a valid definition for the absolute value of a complex number would be

$$|z| = (z\overline{z})^{\frac{1}{2}}.$$

If z and y are complex numbers then

(a) $\overline{z+y} = \overline{z} + \overline{y}$,
(b) $\overline{zy} = \overline{z} \cdot \overline{y}$,
(c) $z + \overline{z} = 2\ \text{Re}\{z\}, \quad z - \overline{z} = 2j\ \text{Im}\{z\}$,
(d) $z\overline{z}$ is real and nonnegative,
(e) $|z| > 0$ accept if $z = 0$ then $|0| = 0$,
(f) $|\overline{z}| = |z|$,
(g) $|zy| = |z||y|$,
(h) $|\text{Re}\{z\}| \leq |z|, \quad |\text{Im}\{z\}| \leq |z|$,
(i) $|z + y| \leq |z| + |y|$.

The last property is known as the *triangle* inequality.

Polar form: We will call $z = a + jb$ the *Cartesian* form of a complex number. A complex number can alternatively be expressed as a vector length with an associated angle, as shown in Fig. 1(b). This is called the **polar form** and written as

$$z = r(\cos\theta + j\sin\theta) \tag{6}$$

The following relationships follow from the definitions:

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta,$$

and

$$r = |z| \qquad \text{and} \qquad \theta = \arctan(b/a).$$

Recall that $\tan(\theta) = \tan(\theta + 2\pi k)$ where k is an integer. Thus θ is not unique since $\arctan(\cdot)$ is *multivalued*. The usual convention is to restrict the angle as

$$-\pi < \theta \leq \pi,$$

what is called the *principal value*.

4 Sequences and series

In discrete-time signal processing we make extensive use of sequences and series. A **sequence** is a succession of mathematical elements, usually real or complex numbers, *indexed* by a countable variable, usually an integer. For example,

$$\ldots, x_{-1}, x_0, x_1, x_2, \ldots, x_p, \ldots$$

is an *infinite* sequence of real numbers indexed by the integer p, while

$$z_5, z_6, \ldots, z_n, \ldots, z_{20}$$

is a *finite* sequence of complex numbers indexed by the integer n. The terms of a sequence may be summed together, and Σ-notation is used to represent this process as

$$\sum_{n=a}^{b} x_n = x_a + x_{a+1} + \cdots + x_{b-1} + x_b \tag{7}$$

where a and b are the limits of the summation and n is the index. Note that the variable n, appearing on the left hand side of (7) but not on the right, is just a *dummy variable* and can be replaced with some other symbol without affecting the significance of the right hand side of (7). The symbology of (7) is called a **series**.

It is desirable to find *closed form solutions* for series whenever possible. A popular technique for doing this is a **telescoping series**, in which a series is resolved into successive terms or parts that cancel. The most common example of a series having a closed form solution is the **geometric series** defined as

$$S = \sum_{n=a}^{b} x^n. \tag{8}$$

The trick for creating a telescoping series is to recognize that the next term of the series can be created by a mathematical operation on the present term. For example in the case of the geometric series the $(n+1)$th term is created from the nth term by simply multiplying by x. Hence,

$$\begin{aligned}
(1-x)S &= (1-x)\sum_{n=a}^{b} x^n \\
&= \sum_{n=a}^{b} x^n(1-x) \\
&= \sum_{n=a}^{b} x^n - x^{n+1} \\
&= \left(x^a - x^{a+1}\right) + \left(x^{a+1} - x^{a+2}\right) + \cdots + \left(x^{b-1} - x^b\right) + \left(x^b - x^{b+1}\right) \\
&= x^a - x^{a+1} + x^{a+1} - x^{a+2} + \cdots + x^{b-1} - x^b + x^b - x^{b+1} \qquad (9) \\
&= x^a - x^{b+1}. \qquad (10)
\end{aligned}$$

The term ‘telescoping’ derives from (9) where successive groups of terms cancel as if linked like a telescope. Dividing the previous result by $(1-x)$ results in the geometric series formula

$$\boxed{\sum_{n=a}^{b} x^n = \frac{x^a - x^{b+1}}{1-x}} \tag{11}$$

provide $x \neq 0$ and $x \neq 1$. The telescoping series concept is not limited to geometric series.

Infinite series are a common occurrence in signals and systems. In this case it is necessary to consider the *convergence* of such a series. For example, the infinite geometric series would take the form

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}} \tag{12}$$

if the series *converges*, which is under conditions of $|x| < 1$. Otherwise the series *diverges*. To see that (12) is true, put $a = 0$ and note

$$\lim_{\substack{n\to\infty \\ |x|<1}} x^n = 0$$

in (11).

Practice: To get a feel for series manipulations consider showing that

$$\sum_{n=a}^{\infty} x^n = \frac{x^a}{1-x} \tag{13}$$

without using a telescoping series or the generalized geometric series formula. Instead use (12).

Solution: Recall that the index of summation is a dummy variable so that it may be replaced with $m = n - a$ (or $n = m + a$) as

$$\begin{aligned}
\sum_{n=a}^{\infty} x^n &= \sum_{m=0}^{\infty} x^{m+a} \\
&= \sum_{m=0}^{\infty} x^a x^m \\
&= x^a \sum_{m=0}^{\infty} x^m \\
&= x^a \left(\frac{1}{1-x}\right) \\
&= \frac{x^a}{1-x}.
\end{aligned}$$

5 Differentiation

Calculus centers mainly on two concepts, differentiation and integration, and the relationship between them. Several good books, such as Larson and Hostetler [2] and Boyce and DiPrima [3], cover these concepts and the mechanics of the operations for a wealth of mathematical functions. The core ideas are presented here to refresh the memory of students who have taken a series of calculus courses.

5.1 The derivative

It is often desirable to consider the rate of change of a function or the slope of a curve. Consider a real function f of a real variable (i.e., $f : \mathbb{R} \to \mathbb{R}$) which is defined on the interval $[a, b]$. Then for any real numbers x and t in the interval $[a, b]$ where $x \neq t$ the **derivative** of a function is defined as a limit as t tends to x of the *difference quotient*, i.e.,

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x} \tag{14}$$

provided the limit exists. A limit process is a rigorous concept beyond the scope of this document. However, in the case of the real line one can think of the limit requiring the function to approach the same quantity from the positive and negative sides of the point x. If f and g are functions differentiable at x then

(a) $(f + g)'(x) = f'(x) + g'(x)$. (Additivity)

(b) $(\alpha f)'(x) = \alpha f'(x)$, $\alpha =$ constant. (Homogeneity)

(c) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$. (Product rule)

(d) $\left(\dfrac{f}{g}\right)'(x) = \dfrac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$, $g(x) \neq 0$. (Quotient rule)

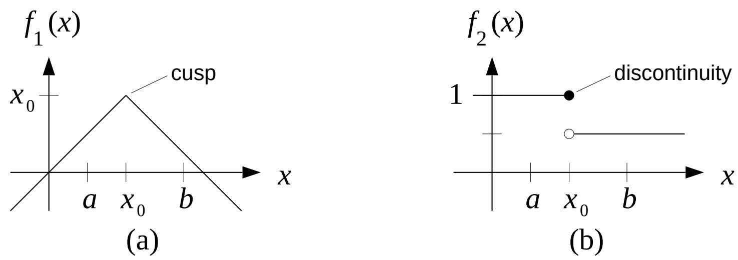

Figure 2: Two functions that are not differentiable at a. The function of (a) has different slopes on either side of the a. The function of (b), although having the same slope on either side of a, is discontinuous at a, precluding it from being differentiable there.

Consider some examples of functions that fail to be differentiable at $x = x_0$. Let the function $f_1 : \mathbb{R} \to \mathbb{R}$ be defined as

$$f_1(x) = \begin{cases} x, & x \leq x_0 \\ 2x_0 - x, & x > x_0. \end{cases} \tag{15}$$

The above form is called *case* representation for the function. The left column is the function's value and the right column is the region or interval over which this value holds true. It is good practice to cover the whole *domain* of the function in the right column without including points of the domain more than once. In the above example, $\{x \leq x_0\} \cup \{x > x_0\} = \mathbb{R}$ and only one of the cases includes the point x_0. Consistency in the right column is also a good idea. In the above example x is always on the left of the inequality symbol for readability. In any case, the function $f_1(x)$, shown in Fig. 2(a), is not differentiable at x_0 because the slope of the function is different on either side of x_0.

Now consider the function $f_2 : \mathbb{R} \to \mathbb{R}$ defined as

$$f_2(x) = \begin{cases} 1, & x \leq x_0 \\ \frac{1}{2}, & x > x_0. \end{cases} \tag{16}$$

Here the slope of $f_2(x)$, shown in Fig. 2(b) is the same on either side of x_0 but the discontinuity at x_0 precludes the function from being differentiable there. Consider

$$\begin{aligned} f'_-(x_0) &\triangleq \lim_{\substack{t \to x_0 \\ t < x_0}} \frac{f(t) - f(x_0)}{t - x_0} = \lim_{\substack{t \to x_0 \\ t < x_0}} \frac{1 - 1}{t - x_0} = 0, \\ f'_+(x_0) &\triangleq \lim_{\substack{t \to x_0 \\ t > x_0}} \frac{f(t) - f(x_0)}{t - x_0} = \lim_{\substack{t \to x_0 \\ t > x_0}} \frac{0 - 1}{t - x_0} = -\infty. \end{aligned} \tag{17}$$

Hence, no valid limit exists for the difference quotient. One should not think of the symbol $-\infty$ in (17) as being a number. Instead this simply represents the concept of *unboundedness* of the limit process.

THEOREM: If $f(x)$ is differentiable at x_0 then $f(x)$ is continuous at x_0.

Proof: We must show that $\lim_{t \to x_0} f(t) = f(x_0)$:

$$\begin{aligned} \lim_{t \to x_0} f(t) &= \lim_{t \to x_0} f(t) - f(x_0) + f(x_0) \\ &= f(x_0) + \lim_{t \to x_0} \frac{f(t) - f(x_0)}{t - x_0} \cdot (t - x_0) \\ &= f(x_0) + f'(x_0) \cdot 0 \\ &= f(x_0). \qquad \square \end{aligned}$$

This theorem asserts that *continuity* of f at x is a *necessary* condition for the existence of the derivative. However, continuity is not a *sufficient* condition. (See Section 11 on Logic.)

Notation: There are three common conventions for writing the derivative of the function $y = f(x)$, Boyce and DiPrima [3]:

$$\begin{aligned} &y',\ f'(x) && \text{(Prime or Lagrange notation)} \\ &Dy,\ Df(x) && \text{(Operator notation)} \\ &\frac{dy}{dx},\ \frac{df}{dx}(x) && \text{(Leibniz notation)} \end{aligned} \tag{18}$$

along with a combined fourth notation

$$\frac{d}{dx}\,y,\ \frac{d}{dx}\,f(x) \quad \text{(Leibniz/operator notation)}$$

It is extremely important to realize that these all represent the same mathematical element, the derivative of the function $f(x)$.

5.2 The differential

The Leibniz notation of (18) is the most intuitive of the accepted conventions because it appears to be in the form of a quotient, thus providing a memory aid for the difference quotient for the formal definition of differentiation. On the other hand, this can also lead to confusion because dy/dx is, in fact, *not* a ratio at all. Students often come to think of the quantities dy and dx as 'infinitesimals' which vanish in such a way that the ratio of the two becomes the derivative. However, Courant and John [4] suggested that imagining dy and dx to be 'infinitesimally small' is mathematically meaningless serving only to obscure an otherwise mathematically sound definition of differentiation. Nonetheless, the notation

$$dy = f'(x)\,dx \tag{19}$$

is used for convenience. (We can get by without it.) Here dy is called a **differential**. The way to think of this notation is to imagine at the point x a small increment dx is made and the change in the function is linearly approximated using the slope $f'(x)$. Note that (19) indicates that the differential is dependent both on x (via $f'(x)$) and dx. If the Lagrange notation of (19) is replaced by the Leibniz notation, i.e.,

$$dy = \frac{dy}{dx}\,dx$$

it becomes evident that dy and dx standing alone is not the same as the letters in the ratio dy/dx. The least confusing way to use dy alone is just as it is defined in (19) and assign no other meaning to it.

5.3 L'Hospital's rule

Often the limit of an expression consisting of the ratio of two differentiable functions which is indeterminate can be found using **L'Hospital's rule**. It is imperative that the conditions under which L'Hospital's rule can be applied be understood to avoid the mistake of improperly using this powerful concept.

L'Hospital's Rule: Given functions f and g differentiable on the open segment (a, b) and g is never zero in this segment, if either

$$\lim_{x\to a^+} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a^+} g(x) = 0, \tag{20}$$

or

$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x\to a^+} g(x) = \pm\infty, \tag{21}$$

where all combinations of plus and minus is allowed in (21), *then* if

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$$

exists, it follows that

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

The formula also holds for x approaching b from the left, i.e., $x \to b^-$.

Example: Consider the indeterminate limit

$$\lim_{x \to 0} \frac{\sin x}{x},$$

and assign $f(x) = \sin x$ and $g(x) = x$. Then (20) holds and by L'Hospital's rule

$$\begin{aligned} \lim_{x \to 0} \frac{\sin x}{x} &= \lim_{x \to 0} \frac{\cos x}{1} \\ &= 1. \end{aligned} \tag{22}$$

6 Integration

There are a variety of ways to define an integral. However, the most common definition used in engineering applications is the Riemann integral. It is the most straightforward of the integrals and corresponds well to applications such as calculating area, volume, torque, stress, and so forth. There are some drawbacks to the Riemann integral. (The most apparent to us will be the use of a not so legitimate *impulse* function to create a *sifting* property.)

6.1 The Riemann integral

The notion of integration is one usually taught in relation to finding the area under a curve by breaking up, or *partitioning*, the interval of integration into smaller and smaller subintervals that tend toward zero. More formally, a **partition** of interval $[a, b]$ is a finite set of points for which $a = x_0 < x_1 < \cdots < x_n = b$. The length of each subinterval is defined as

$$\Delta x_i = x_i - x_{i-1}, \qquad i = 1, 2, \ldots, n.$$

This partitioning concept is denoted

$$\Delta = \{x_0, x_1, \cdots, x_n, \}$$

and define the **norm** of the partition as

$$||\Delta|| = \max\ \Delta x_i, \qquad i = 1, 2, \ldots, n.$$

Let x_i^* be an intermediate value in subinterval i.

Definition: If f is a bounded function on the interval [a,b] then

$$\int_a^b f(x)\, dx = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i, \tag{23}$$

provided the limit exist is the **Reimann integral** of f from a to b.

Again, the proper limit *must* exist, in which case f is called integrable on [a,b]. Otherwise, f is not integrable there.

6.2 Fundamental theorem of calculus

Although the definition of (23) can be used to find closed form solutions to simple integral equations, evaluation of the integral is generally done using an antiderivative. A function $F(x)$ is said to be an **antiderivative** of the function $f(x)$ if $F'(x) = f(x)$. The *fundamental theorem of calculus* relates integration and differentiation.

FUNDAMENTAL THEOREM OF CALCULUS: Given the function f which is bounded on the closed interval $[a, b]$ and continuous on the open interval (a, b), if F is a continuous function on $[a, b]$ such that F is the antiderivative of f on the segment (a, b) then

$$\begin{aligned}\int_a^b f(x)\,dx &= F(b) - F(a) \\ &\triangleq F(x)\big|_a^b .\end{aligned}$$

This theorem becomes so natural to the student that it is often overlooked and the result treated as though it were the definition for an integral, which it is not. It is important to remember the necessary hypotheses to avoid applying the theorem improperly. For example, consider the following question.

Question: Are the functions of Fig. 2 integrable on the interval $[a, b]$?

Answer: Yes, they are integrable on the interval [a,b] because the limit of (23) exists. However, the fundamental theorem of calculus does not apply in either case. Neither of the functions in Fig. 2 have an antiderivative which is continuous at x_0. Furthermore, the function of Fig. 2(b) is discontinuous at x_0 which, again, precludes the application of the fundamental theorem of calculus. Hence, it is necessary to apply the fundamental theorem of calculus in a piecewise fashion as

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^{x_0} f(x)\,dx + \int_{x_0}^b f(x)\,dx \\ &= F_{x_0}^-\big|_a^{x_0} + F_{x_0}^+\big|_{x_0}^b ,\end{aligned}$$

where $F_{x_0}^-$ and $F_{x_0}^+$ are the appropriate antiderivatives on the negative and positive side of x_0, respectively.

6.3 Integration by parts

One of the more powerful techniques for evaluating integrals is **by parts**. This technique is based simply upon the product rule of differentiation. Recall the product rule

$$(uv)'(x) = u(x)v'(x) + v(x)u'(x),$$

or subtracting $v(x)u'(x)$ from both sides

$$u(x)v'(x) = (uv)'(x) - v(x)u'(x).$$

Now integrate to get

$$\boxed{\int_a^b u(x)v'(x)\,dx = (uv)(x)\big|_a^b - \int_a^b v(x)u'(x)\,dx.} \tag{24}$$

If we make use of the differential notation of (19), it is easy to remember integration by parts for the *indefinite* integral. Equation (24) becomes

$$\int u\,dv = uv - \int v\,du.$$

Remember the product rule differential form, $d(uv) = u\,dv + v\,du$, and the formula should easily come to mind.

Example: Consider evaluating

$$\int_a^b xe^x\,dx.$$

Since $d(e^x)/dx = e^x$ it is wise to chose e^x as the differential portion of the parts formula. That is

$$\begin{aligned}\int_a^b \underbrace{x}_{u}\underbrace{e^x\,dx}_{dv} &= \underbrace{xe^x}_{uv}\Bigg|_a^b - \int_a^b \underbrace{e^x}_{v}\;\underbrace{dx}_{du} \\ &= (xe^x - e^x)\big|_a^b .\end{aligned}$$

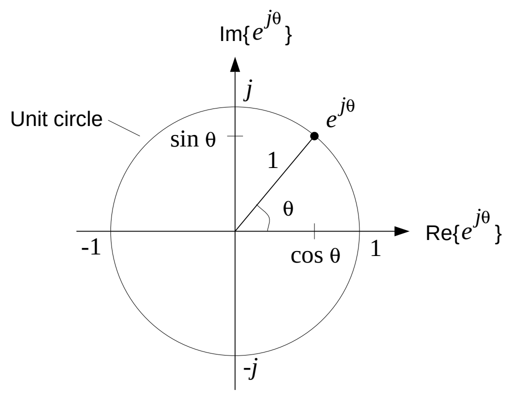

Figure 3: The unit complex exponential $e^{j\theta}$ in the z-plane. It can be thought of as a unit vector at a Cartesian angle θ, lying on what is termed the *unit circle*.

7 Complex exponential

In the integration by parts example we made use of the property of the real exponential that $d(e^x)/dx = e^x$ where x is a real number. It is natural to extend this same idea such that $d(e^z)/dz = e^z$ is true, where z is now a complex number. If $z = a + jb$, then similar to the real exponential we would like

$$e^z = e^{a+jb} = e^a e^{jb},$$

where e^a is the conventional real exponential. Differentiating the above expression with respect to z is beyond the scope of this paper. Suffice it to say, what is needed to complete the definition of the complex exponential is a function for which

$$\frac{d(e^{jb})}{db} = j\, e^{jb}. \tag{25}$$

The equation which does this is the powerful **Euler's formula**

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta.} \tag{26}$$

The reader should verify that this, in fact, does satisfy (25). Figure 3 shows the relationship of Euler's formula as a vector in the complex plane. Note that the vector is of unit length.

With the aid of Euler's formula applied to (6), a third possible representation for a complex number is

$$z = r\, e^{j\theta},$$

where r and θ are defined as for the polar representation. Note that $\overline{z} = r\, e^{-j\theta}$.

8 Taylor's theorem and series

Many nonpolynomial functions can be represented by a series whose coefficients are derived from the derivative of the function. **Taylor's theorem** forms the heart of this theory.

TAYLOR'S THEOREM: If a function $f(t)$ has $n-1$ continuous derivatives on the closed interval $[a, b]$ and the nth derivative, $f^{(n)}$(t), exist on the open interval (a, b), let x and c be two distinct points in $[a, b]$. Then there exists a point ξ between x and c such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \underbrace{\frac{f^{(n)}(\xi)}{n!}(x-c)^n}_{\text{remainder}}. \tag{27}$$

The important consequence of this theorem is that if the remainder term of (27) tends to zero for a particular function in some region about c then the function can be written as a **Taylor series**

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k,} \tag{28}$$

valid in that region. This is said to be expanded about c. If $c = 0$ then (28) is called a **Maclaurin series**.

There are three very important functions for which the remainder term tends to zero for all x when $c = 0$. These should be derived and recognized, if not memorized, by the student. The functions are

(a) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad -\infty < x < \infty.$

(b) $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad -\infty < x < \infty.$

(c) $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad -\infty < x < \infty.$

Practice: Evaluate

$$\lim_{x \to 0} \frac{\sin x}{x}$$

without using L'Hospital's rule.

Solution: Use a Maclaurin series as

$$\begin{aligned} \lim_{x \to 0} \frac{\sin x}{x} &= \lim_{x \to 0} \frac{1}{x} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \\ &= \lim_{x \to 0} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!} \\ &= \lim_{x \to 0} \frac{(-1)^0 x^0}{1!} \\ &= \lim_{x \to 0} 1 \\ &= 1, \end{aligned} \tag{29}$$

which agrees with (22). Here, all but the $k = 0$ term vanished in (29).

Reader problem: Use the Maclaurin series' above to show Euler's formula, $e^{j\theta} = \cos\theta + j\sin\theta$.

9 Partial fraction expansion

See the appendix of Haykin and Van Veen for a discussion of partial fraction expansion.

10 Mathematical induction

Often with series it is convenient to prove or show the validity of a closed form solution using a technique called **mathematical induction**. This procedure is actually a sequence of implications. That is, begin by showing the validity of proposition $P1$ then use induction to show that proposition Pn implies proposition $P(n+1)$. That is

$$P1 \Rightarrow P2 \Rightarrow P3 \Rightarrow \cdots$$

where the symbol '$\Rightarrow$' means implies. Note that the implication is only in one direction and that it is imperative to establish the validity of proposition $P1$ in order to start the induction.

The advantage of mathematical induction is that a guess can be made at the closed form solution then shown to be true. Granted, guessing is not an efficient manner of finding a solution. Nonetheless, we are sometimes left with little option other than using induction if alternate techniques are too cumbersome or are nonexistent as of yet. As an example, reconsider the closed form solution for the geometric series. It was derived previously using the telescoping technique. To show that

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x} \tag{30}$$

is, in fact, correct without deducing the solution, use induction as follows.

$P1$: Let $n = 0$ in which case (30) becomes

$$\begin{aligned} \frac{1 - x^1}{1 - x} &= 1 \\ &= x^0 \end{aligned}$$

which is the only term in the series when $n = 0$.

$Pn \Rightarrow P(n+1)$: Assume that (30) is true. Then

$$\begin{aligned} \sum_{i=0}^{n+1} x^i &= \frac{1 - x^{n+1}}{1 - x} + x^{n+1} \\ &= \frac{1 - x^{n+1}}{1 - x} + \frac{x^{n+1}(1 - x)}{1 - x} \\ &= \frac{1 - x^{n+1} + x^{n+1} - x^{n+2}}{1 - x} \\ &= \frac{1 - x^{n+2}}{1 - x}. \end{aligned}$$

which agrees with the formula in (30) if n is replaced by $n + 1$. Hence, (30) is, in fact, correct.

11 Logic

In mathematical reasoning we often consider *negating* the logic of a statement. People often do this naturally without thinking much about the logic of the process. To be precise, say that statement $S1$ implies statement $S2$ has been shown to be true. That is

$$S1 \Rightarrow S2. \tag{31}$$

If the statements are negated and the direction of implication is reversed we have what is called the **contrapositive**. That is

$$\overline{S1} \Leftarrow \overline{S2}.$$

For example, it is known that if a function is differentiable at a point then the function is also continuous at that point. The following logic illustrates the concept of contrapositive.

THEOREM: If $f(x)$ is differentiable at a then $f(x)$ is continuous at a.

Contrapositive: If $f(x)$ is not continuous at a then $f(x)$ is not differentiable at a.

Recall that previously these facts were stated something like continuity is a necessary condition for differentiability. Return to (31) and consider the following statements: $S2$ is necessary for $S1$; $S1$ is sufficient for $S2$. These two claims express the same idea. The reader should now verify that the logic

$$S1 \Longleftrightarrow S2$$

translates to $S1$ is a necessary and sufficient condition for $S2$.

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