ELECTRICAL ENGINEER'S TIP TO HKCEE/HKALE (SYSTEMS)

Hoi Wong (wonghoi.ee@gmail.com)

B.S.E.E., Math (Hons.)
University of Wisconsin -Madison

ABSTRACT

While I was in college, I realized that the Maths and Physics curriculum in Hongkong presents HKCEE/HKALE materials in an inefficient manner. Through this short article, I share my EE toolbox in mathematics and electronics to fellow inquisitive or fustrated students.

This article serves as an extension to secondary school curriculum, plus an alternative introduction to electrical circuits. With the help of these new perspectives, I hope students will find the topics easier than expected. I plan to be brief to save readers' time.

1. SYMBOLS

1.1. Mathematical Symbols

$\mathbb{R}$	The real line
$\mathbb{C}$	The complex plane
$\mathbb{Z}$	The set of integers
A	a matrix
$\hat{j}$	$\sqrt{-1}$
z, w	complex variables
θ	argument of complex number
ω	angular frequency, $\omega = 2\pi f$
$\in$	belonging to
$\forall$	for all
$a, b \in \mathbb{R}$	a, b are real numbers
$z \in \mathbb{C}$	z is a complex number
$n, k \in \mathbb{Z}$	n, k are integers
L	inductance
C	capacitance
X	reactance, $X \in \mathbb{R}$
S	complex power, $\mathbf{S} \in \mathbb{C}$
Z	impedance, $\mathbf{Z} \in \mathbb{C}$
Ph(**Z**)	phase of **Z** in a.c. circuits $\equiv$ arg(**Z**)

2. VECTORS

[Additional Mathematics, Physics (CE/AL)]

1. Learn the unit vector notation:

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} \tag{1}$$

Unit vectors always have magnitude of 1. $|\hat{u}| \equiv 1$.

2. When doing mechanics problem, resolve your vectors into two **perpendicular**[1] components pointing along *x*-axis and *y*-axis. Since $\hat{x}$ and $\hat{y}$ components are NOT related[2], they can be computed seperately and combined (added) in the final answer.

3. COMPLEX NUMBERS

[Pure Maths, AL-Phyics, Trigonometic Identities]
Prerequisite: Vectors, Co-geom, Trigo, Pyth Thm.

Most of the fruitful tricks in this article requires high proficiency in complex numbers $\mathbb{C}$. People who underestimate their familiary with complex numbers will suffer in topics founded on it.

In this article, it's my job to convince you that complex number doesn't make your life more complicated, *given you need to get the same job done*, but to simplify your work.

Due to the imaginary nature of complex numbers, I'll introduce complex numbers from scratch to make you feel a bit more comfortable than 'being told' by your textbook.

NOTE: for those who have seen complex numbers, you will notice that I use $\hat{j}$ instead of *i*. *j* is electrical engineers' notation to avoid confusion with electric current, and adding a hat is my own notation, which I will explain in the coming section.

3.1. Introduction

You may skip this section if you are willing to accept the idea of complex numbers without explanation.

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[1] or 'orthorgonal' in university language

[2] a result of orthorgonality: $\hat{x}, \hat{y}$ component doesn't affect each other

In the mathematics you've been studying till now, you've been assuming *real* values such as -34, 4.87, π, 0, etc. They can be drawn on a line[3], which is called the *real line*, $\mathbb{R}$. However, you cannot find a root when you have to take the square-root of a negative number in real number.

Mathematicians introduced a symbol $\hat{j}$ to denote the problematic case of $\sqrt{-1}$. As we know $\hat{j}$, doesn't fit into the $\mathbb{R}$, they decided to put $\hat{j}$ into another dimension[4] and call it *imaginary number*. Now, we have two types of numbers, real and imaginary working *together*. One dimension gives a line, and two dimensions gives a plane. So, complex numbers lives on a plane $\mathbb{C}$, called the Argand plane.

After years of work, mathematicians consolidated $\mathbb{C}$ as an extension to $\mathbb{R}$, i.e. $\mathbb{R} \subset \mathbb{C}$. Complex numbers enables users of mathematics to chip in intermediate calculations/steps that doesn't have a physical explaination, yet arriving at the correct answer at the end of the day, with less fuss.

3.2. Definition and Vector Analogy

$$\hat{j}^2 \equiv -1 \qquad (2)$$

The more common form is $\hat{j} \equiv \sqrt{-1}$.

In this section, I'll show you that complex numbers are very analogus to vectors in 2D co-geom. *Skip this section if you accept this fact.*
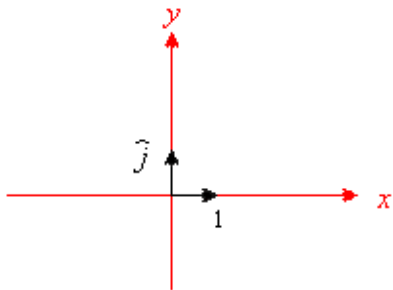
Consider a vector $\vec{z}$

$$\vec{z} = \alpha\vec{i} + \beta\vec{j} \qquad (3)$$

For convenience, we prefer i and j to be unit vectors so we just have to care about the 'direction' when we use them. α and β will be scaled appropriately to a and b by equation (1):

$$\vec{z} = a\hat{i} + b\hat{j} \qquad (4)$$

To get neat numbers, we assume $\hat{i}$ to lying x-axis and $\hat{j}$ lying on y-axis, and hence they are **perpendicular** to each other.

Complex number is not officially a vector, so we have to

[3]You saw that in inequalities
[4]more formally, domain

remove the top arrow from $\vec{z}$. We also remove $\hat{i}$ because we don't need a special symbol for real numbers. Keep in mind that $\hat{j}$ is now an imaginary number, not a vector yet it works like a vector. Then we have

$$z = a + b\hat{j} \qquad (5)$$

The symbol $\hat{j}$ have many advantages:
1. Avoid confusion with electrical current i
2. The hat reminds me of the unit vector analogy.
3. It coincides with the vertical vector in cartesian plane.
4. i and j are often used as indicies in summation.

Mind that most complex numbers calculations are done by treating them as vectors, and drawing them on Argand diagram. Once the diagram is drawn, all information can be extracted from it with much ease.

3.3. Rectangular Representation

You just saw the rectangular form of complex number z:

$$z = a + b\hat{j}$$

a is the real part of z, denoted by $\text{Re}(z)$
b is the imaginary **part** of z, denoted by $\text{Im}(z)$

It is very important to realize that $a, b \in \mathbb{R}$. The only imaginary number in z is $\hat{j}$. The purpose of a is to scale the real number 1, and the purpose of b is to scale the imaginary number $\hat{j}$.

Rectangular form is easy to add and subtract, however, it is tedious to multiply. We have polar form that will make multiplication as easy as real numbers.

3.4. Polar Representation

Once you put things into a coordinate plane, Pythegora's theorem and your basic trigonometry kicks in like 2D co-geom.
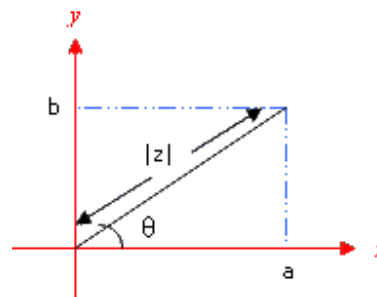
Instead of expressing in terms of a and b, the diagram revealed the possibility to express z in terms of $|z|$ and θ.

$|z|$: the magintude[5] of z
θ: the argument[6]of z, written as arg(z).

One of the misfortunes of education in Hongkong is that they keep the Euler number $e \approx 2.71828...$ from students until Form 7, depriving their rights to take full advantage of laws of indicies.

The polar form of z is written as

$$z = |z|e^{\hat{j}\theta} \qquad \text{where } \theta = \arg(z) \qquad (6)$$

To convert from rectangular form to polar form, you can read the magnitude $|z|$ with your knowledge in Pythagora's Theorem and the argument arg(z) from the basic trigonometry you learnt in Form 2:

$$|z| = \sqrt{a^2 + b^2} \qquad (7)$$

$$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right) \qquad (8)$$

De Moivre's Theorem makes polar coordinates ideal for multiplication and division of complex numbers. It says:

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \qquad (9)$$

With the introduction of e, the freaky M.I. proof in Pure Math is now shortened to the kind of algebra doable by a Form 2 boy:

$$z = z_1 z_2 = \left(|z_1|e^{\hat{j}\arg(z_1)}\right)\left(|z_2|e^{\hat{j}\arg(z_2)}\right) \qquad (10)$$

$$= (|z_1||z_2|)e^{\hat{j}[\arg(z_1)+\arg(z_2)]} = |z|e^{\hat{j}\arg(z)} \qquad (11)$$

From (11), we obtained $|z| = |z_1||z_2|$ and equation (9).

If possible, do addition and subtraction in rectangular form and do multiplication and division in polar form.

3.5. Conjugates

The conjugate of z, written as $\bar{z}$, means flipping the point z along the real axis (i.e. upside down) in Argand diagram.

In rectangular form, conjugate flips the sign of the imaginary part, Im(z):

$$z = a + b\hat{j} \quad \Leftrightarrow \quad \bar{z} = a - b\hat{j} \qquad (12)$$

In polar form, conjugate flips the sign of the phase angle, or the argument, arg(z):

$$z = |z|e^{\hat{j}\theta} \quad \Leftrightarrow \quad \bar{z} = |z|e^{-\hat{j}\theta} \qquad (13)$$

Roots to an equation always comes in conjugate pairs. e.g. if $3 + 4\hat{j}$ is a root, $3 - 4\hat{j}$ must be the other root of the equation. This will save you from repeating similar steps when

solving ODE (good news to Applied Math folks).

Here's a weak explanation: we knew

$$(+\hat{j})^2 = -1$$
$$(-\hat{j})^2 = -1$$

but there is no way to tell whether the root to an equation is on the upper half plane or the lower half plane. By symmetry, both possibilities are the roots. A very similar situation to solving $a^2 = 9$: the roots are 3 and -3.

3.6. Euler's formula

The curriculum in Hongkong also mentions the notation cis(θ) as a way to write the polar form. However, it is a result of Euler's formula in this section.

We've discussed how to convert from rectangular form to polar form. Given polar form, we convert it back to rectangular form using argand diagram and a bit of basic trigonometry:
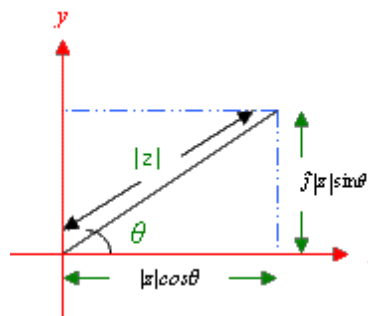
From the diagram, the converted rectangular form is:

$$z = |z|\cos\theta + \hat{j}|z|\sin\theta$$

Grouping terms and equating to polar form,

$$z = |z|e^{\hat{j}\theta} = |z|(\cos\theta + \hat{j}\sin\theta)$$

Simplifying we get the famous Euler's formula:

$$e^{\hat{j}\theta} = \cos\theta + \hat{j}\sin\theta \qquad (14)$$

It is very important to notice:

$$|e^{\hat{j}\theta}| \equiv \sqrt{cos^2\theta + sin^2\theta} \equiv 1 \qquad (15)$$

Now you know *cis* stands[7] for cosine+*i*sine.

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[7]i is the official symbol for $\sqrt{-1}$

3.7. Properties of Complex Numbers

After going through rudimentary definitions, here comes the properties of complex numbers which you need to remember by heart and use them often as shortcuts:

$$|z| = z\bar{z} \tag{16}$$
$$|zw| = |z||w|^2 \tag{17}$$
$$\overline{z+w} = \bar{z} + \bar{w} \tag{18}$$
$$\overline{zw} = \bar{z}\,\bar{w} \tag{19}$$
$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \tag{20}$$
$$\mathrm{Im}(z) = \frac{z - \bar{z}}{2\hat{j}} \tag{21}$$
$$|z + w| \leq |z| + |w| \tag{22}$$

Equation (22) is the famous 'triangle inequality' in $\mathbb{C}$.

Knowing the hypotenuse $|z|$ is the longest side of the right angle triangle with legs $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$,

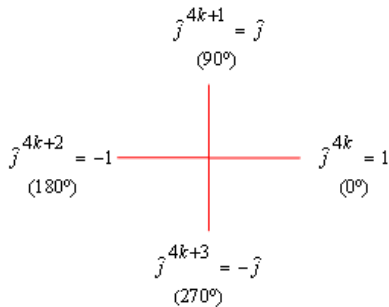$$\mathrm{Im}(z) < |z| \tag{23}$$
$$\mathrm{Re}(z) < |z| \tag{24}$$

3.8. Miscellaneous

Exam writers love to test your understanding of Argand diagram by asking something like the locus of $|z - (a + b\hat{j})| = r$. If you keep the length fixed and allow the phase angle to swril, you get a circle of radius r centered [8] on $a + b\hat{j}$.

Note that $\hat{j}$ serves as a $90°$ rotator when it's multiplied to a complex number. $\forall k \in \mathbb{Z}$,

4. TRIGONOMETRIC IDENTITIES

Though not all trigonometric identities can be proved neatly in complex numbers, there are cases where complex numbers can save you from reading mind-numbling geometrical explanations.

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[8] You already know the 1D version: $f(x - \delta)$ means shifting the '0' point to δ. It's just an extension to 2D

Euler's equation (14) showed the complex exponential $e^{\hat{j}\theta}$ is actually a compact package of sines and cosines. using (20) and (21), we obtain

$$\mathrm{Re}(e^{\hat{j}\theta}) = \cos\theta = \frac{e^{\hat{j}\theta} + e^{-\hat{j}\theta}}{2} \tag{25}$$

$$\mathrm{Im}(e^{\hat{j}\theta}) = \sin\theta = \frac{e^{\hat{j}\theta} - e^{-\hat{j}\theta}}{2\hat{j}} \tag{26}$$

Example. Proof product to sum formula:

$$\cos\frac{\alpha - \beta}{2} \sin\frac{\alpha + \beta}{2} = \frac{1}{2}(\sin\alpha + \sin\beta) \tag{27}$$

Proof.

$$\cos\frac{\alpha - \beta}{2} \sin\frac{\alpha + \beta}{2}$$

$$= \left(\frac{e^{\hat{j}\frac{\alpha-\beta}{2}} + e^{-\hat{j}\frac{\alpha-\beta}{2}}}{2}\right)\left(\frac{e^{\hat{j}\frac{\alpha+\beta}{2}} + e^{-\hat{j}\frac{\alpha+\beta}{2}}}{2\hat{j}}\right)$$

$$= \frac{1}{4\hat{j}}\left(e^{\hat{j}\frac{2\alpha}{2}} - e^{-\hat{j}\frac{2\alpha}{2}} + e^{\hat{j}\frac{2\beta}{2}} - e^{-\hat{j}\frac{2\beta}{2}}\right)$$

$$= \frac{1}{2}\left[\left(\frac{e^{\hat{j}\alpha} - e^{-\hat{j}\alpha}}{2\hat{j}}\right) + \left(\frac{e^{\hat{j}\beta} - e^{-\hat{j}\beta}}{2\hat{j}}\right)\right]$$

$$= \frac{1}{2}(\sin\alpha + \sin\beta)$$

5. INVERTING MATRICES

`[Pure Mathematics]`

Gaussian elimination and row echlon form is taught in Pure Maths. However, old textbooks didn't exploit the following simple definition to invert matrices by Gaussian Elimination:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \tag{28}$$

The same way as you solve a system of equations in Pure Maths, the above equation can solved for $\mathbf{A}^{-1}$ by computing

$$\mathrm{rref}[\mathbf{A}|\mathbf{I}] = [\mathbf{I}|\mathbf{A}^{-1}] \tag{29}$$

and $\mathbf{A}^{-1}$ is the inverted matrix you need.

The following example might convince you that this method is less painful than the cofactor method in old texts.

Example. Consider inverting a matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 2 \\ 4 & 6 & 8 \end{pmatrix}$$

Now, let's compute rref[$\mathbf{A}|\mathbf{I}$]:

$$
\left[\begin{array}{ccc|ccc}
1 & 3 & 5 & 1 & 0 & 0 \\
7 & 9 & 2 & 0 & 1 & 0 \\
4 & 6 & 8 & 0 & 0 & 1
\end{array}\right]
$$

$$
\rightarrow
\left[\begin{array}{ccc|ccc}
1 & 3 & 5 & 1 & 0 & 0 \\
0 & 6 & 2 & 4 & -1 & -1 \\
0 & 12 & 33 & 7 & -1 & 0
\end{array}\right]
\begin{array}{l}
① \times 4 - ③ \\
① \times 7 - ②
\end{array}
$$

$$
\rightarrow
\left[\begin{array}{ccc|ccc}
1 & 0 & -1 & -1 & 0 & 0.5 \\
0 & 0 & -9 & 1 & 1 & -2 \\
0 & 3 & 6 & 2 & 0 & -0.5
\end{array}\right]
\begin{array}{l}
① - ② \times 0.5 \\
② \times 2 - ③
\end{array}
$$

$$
\rightarrow
\left[\begin{array}{ccc|ccc}
1 & 0 & -1 & -1 & 0 & 0.5 \\
0 & 3 & 0 & \frac{8}{3} & \frac{2}{3} & -\frac{11}{6} \\
0 & 0 & 1 & -\frac{1}{9} & -\frac{1}{9} & +\frac{2}{9}
\end{array}\right]
\begin{array}{l}
③ + ② \times \frac{2}{3} \\
② \times -\frac{1}{9}
\end{array}
$$

$$
\rightarrow
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & -\frac{10}{9} & -\frac{1}{9} & \frac{13}{18} \\
0 & 1 & 0 & \frac{8}{9} & \frac{2}{9} & -\frac{11}{18} \\
0 & 0 & 1 & -\frac{1}{9} & -\frac{1}{9} & +\frac{2}{9}
\end{array}\right]
\begin{array}{l}
① \times 3 \\
② \times \frac{1}{3}
\end{array}
$$

$$
\mathbf{A}^{-1} = \frac{1}{18}\begin{pmatrix}
-20 & -2 & 13 \\
16 & 4 & -11 \\
-2 & -2 & +4
\end{pmatrix}
$$

6. ELECTRIC CIRCUITS

[Physics]

Circuit topology is the most common cause for students to hate circuits. This section attempt to help students to overcome the hurdle by introducing the electrical engineer's way of understanding circuits.

Secondary school physics texts, for simplicity, teach circuits by giving circuit patterns to students for memorization. However, it is fatal if students carry this kind of thinking when they start to deal with real circuits. Please be prepared to rip the prespective in those text and start over here.

NOTE: As there are too many diagrams to draw, I chose not to develop this section too much before there's incentive for publishing. If you're hopelessly stuck, contact me for paper and pencil tutoring.

6.1. Philosophy and Practices

1. Electrical engineers always keep in mind that every naming is arbitrary. However, we must be consistent with what we defined in the beginning. For example, you can create a new language that call your male parent 'mom' and your female parent 'dad', as long as you use them *consistently*. This kind of vision is also useful in spotting symmetry and thus saving us tons of work.

2. Also, we treat everything with reference (respect or relative) to a set rules we defined, such as taking an arbitrary point in a circuit and call it ground. Most mistakes comes from assuming incorrect 'concensus' between engineers or forgetting your own set of rules. Therefore, it is a good practice to make clear of what rules you are following and keep track of their consistency.

3. To be helpful to yourself and other people, try to stick to some commonly used convention (though it's grey area) like using red wire for positive terminal and black wire for negative terminal. Be flexible if necessary!

4. Topology is NOT changed by stretching and bending, so regardless of how you extend[9] and bend the wires, the circuit workings remains the same. However, if you disconnect a wire or connect it to another joint in the circuit, the circuit might work differently (e.g. disconnecting a battery lead turns your walkman off.)

6.2. Series and Parallel Topology

Complicated circuits often have series and parallel 'subcircuits' in it. It is important to identify them in a topological point of view.

Series: Not branched[10] at any node/point.
Parallel: Branched at a node/point.

Most people find series topolgy trivial, however, parallel topology is easy too if you cared to find out *all* possible branches when you see a 'parallel' node.
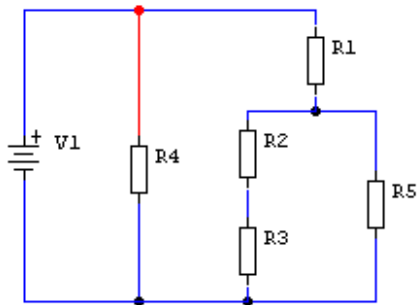
Exercise. Find total resistance seen from voltage source (battery) V:
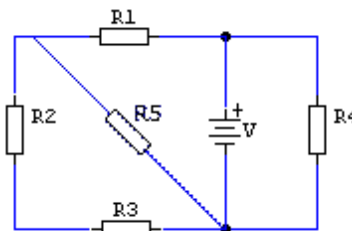
Solution. Tracing *every* possible branches (or routes) and bend the circuit in 3D mind, we can redraw the circuit:

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[9]Where electromagnatic problems are not dominant
[10]split in two or more paths

$$R = R_4//[R_1 + R_5//(R_2 + R_3)]$$

6.3. Voltage, Current and Ground

Due to ambiguity in terminology in electronics literature, some students get confused when they read verbal circuit descriptions. I'll discuss these common misconceptions in this section.

The other name for voltage is 'potential difference' (p.d.). From kindergarten mathematics, we know that it takes two number to 'make a difference', so as p.d.. The sentence structure for voltage is:

$$V_{ab} = \text{"The voltage across A and B"}$$

It is not accurate to say "voltage at node A" *alone* because it's not a finished sentence. The unspoken part of the sentence is "... with respect to the ground ...".

To save breath describing circuits, people pick a node where every measurement is taken with respect/reference to. The node is then called 'Ground'. So, when we try to express 'voltage *at* node' on circuit diagrams, we **must** make clear where the Ground is by drawing a symbol at the reference node.

Ground is quite an arbitrary notation. Despite the 'physical reality', Ground is *always* defined to have zero voltage. For simple circuits with one battery of voltage V_{+-}, most people call the negative terminal ground ($V_- = 0$), and the positive terminal have the voltage of the battery (V_+).

Confusion about voltage values roots from sloppiness: people omit the ground symbol and let readers guess. The battery example was made clear by:

$$V_{+-} = V_+ - V_- = V_+ - 0 = V_+$$

For the sake of completeness, the preprosition for current is *'through'*. e.g. 'Current through node A'

6.4. Mathematical Shortcuts

1. **Parallel impedance**: most scientific calculator a reciporical button[11] so you can calculate parallel impedances by:

$$R_{//} = (R_1^{-1} + R_2^{-1} + R_3^{-1} + ...)^{-1} \qquad (30)$$

2. **Identical Resistors in Parallel**[12]: Divide one of the identical resistance by the number of branches give you the combined resistance. Given n branches with resistance R in each branch, equation (30) can be written as

$$R_{//} = (nR^{-1})^{-1} = \frac{R}{n} \qquad (31)$$

3. **Potential divider**: not all teachers share electronic technician or engineers' vision about potential dividers. You can tell if your physics teacher give you this combined formula *immediately* when he start potential dividers:

$$V_{R_2} = V\left(\frac{R_2}{R_1 + R_2}\right) \qquad (32)$$

If everybody have this formula, do you think HKEA will let you use it directly without intermediate steps? I don't think so. Equation (32) is a consequence of the more primitive form of potential divider:

$$\frac{V_{R_1}}{V_{R_2}} = \frac{R_1}{R_2} \qquad (33)$$

Equation (33) is a more useful and intuitive form. I'll give an example for MC Questions (i.e. you can do it very quickly if you spot nice numbers):

Example 1. Consider a two resistors, $R_1 = 3\Omega$ and $R_2 = 6\Omega$ connected to a 9V battery[13]. What is the voltage across R_2?

In voltage dividers, we only care about the ratio between series resistors. From equation (33), we know $V_{R_1} : V_{R_2} = 3 : 6$. However, we have nice numbers[14] in this problem. Because 3+6=9, 9V is *shared* by two resistors in with 3 parts to R_1 and 6 parts to R_2. Trivially, $V_{R_2} = 6V$

If you ignored the nice numbers, equation (32) might not slow you down because you're already too slow. The next example will show you the full advantage of equation (33).

Example 2. Consider $R_1 = 3\Omega$ is connected with $R_2 = 6\Omega$ in series and the voltage across R_2 is 8V, what is the voltage across R_1?

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[11] [1/x] on Casio, and [x^{-1}] in Sharp
[12] in HKCEE Physics MC
[13] This problem doesn't care about the sign in order to save a diagram
[14] like most MC problems that you thought of as 'time-consumers'

If you use equation (32), you will end up doing algebra to break it down to equation (33). Let's use equation (33):

$$\frac{V_{R_1}}{8} = \frac{3}{6}$$

$$V_{R_1} = 4$$

It's possible to ask for the voltage across both resistors. Nonetheless, it is still easier to find the voltage across R_1 and add to the given voltage.

4. **(Generalized) Current Divider Formula**: Consider this circuit with I given and branch currents I_1 and I_2 to be found:
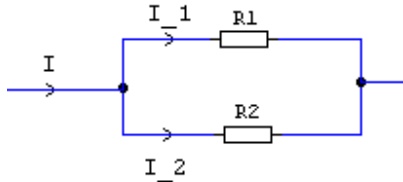
People with electrical intuition will see without deriving,

$$I_1 = I\left(\frac{R_2}{R_1 + R_2}\right) \tag{34}$$

I saw people calculating the parallel branch resistance $R_{/\!/}$ for $V_{/\!/} = IR_{/\!/}$, then have $I_1 = \frac{V_{/\!/}}{R}$. They're actually going through the derivation[15] of my 'Generealized Current Divider Formula'. At branch b

$$I_b = I\left(\frac{R_{/\!/}}{R_b}\right) \tag{35}$$

Proof. Since all the branches share the same pair of node,

$$V_{/\!/} = V_{R_1} = V_{R_2} = ... = V_{R_b} \tag{36}$$

Using Ohms law,

$$I_{/\!/}R_{/\!/} = I_{R_b}R_b \tag{37}$$

which is a cleaner form of (35).

For multiple branches, you don't have to use equation (34) and cascade them. Use equation (35) with $R_{/\!/}$ calculated by equation (30).

6.5. Kirchoff Laws

[Physics (AL/AS)]

Circuits with multiple sources (voltage or current) and bridges often cannot be analzyed by simple series-parallel techniques, therefore a more general approach is needed. KCL and

KVL are trivial ideas strung into a bunch of simultaneous equation for you to solve. Signs in KCL and KVL are typically confusing, and my favorite sign notations are:

1. The point where current start flushing **into** resistances or impedances are defined to be +.

2. KCL: left hand side of the equation is is current in, the right hand side is current out.

Consistently sticking to one set of notations will give you less chance for errors.

7. AC CIRCUITS

[Physics (AS/AL), Electronics (AS)]

In HKALE Physics and Electronics, you *were* supposed to memorize a lot of confusing and messy formula with arctan and square root of sum of squares, plus stupid rules like voltage lead current, etc. It's simply due to Physics community's culture to make sense of electric circuits from real numbers only. Let's see what electrical engineers do in this section!

7.1. Complex Impedance

From now on, when dealing with AC circuits, forget the shit in your textbook (but remember $\omega = 2\pi f$) and treat every capacitor and inductor the same way as you treat resistors. The painful days of phase vectors diagrams, multiple plots of sine wave, damn rotating vectors, etc are over[16]. The new term for this 'virtual+reality resistance' is a complex number called *impedance*, denoted by $\mathbf{Z}$:

$$\mathbf{Z}_{\text{capacitor}} = \frac{1}{\hat{j}\omega C} \tag{38}$$

$$\mathbf{Z}_{\text{inductor}} = \hat{j}\omega L \tag{39}$$

Relating to reactances in your textbook,

$$\mathbf{Z}_{\text{capacitor}} = -\hat{j}X_{\text{capacitor}} \tag{40}$$

$$\mathbf{Z}_{\text{inductor}} = \hat{j}X_{\text{inductor}} \tag{41}$$

I personally don't use the term reactance often.

Complex numbers gives amplitude (magnitude) and phase (argument) information about all quantities in a.c. circuits. So, everything is in a convenient package for manipulations like regular algebra.

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[15]With voltage calculated as by-product

[16]I am glad that I went to America before starting a.c. circuits in Form 7

If your knowledge in the real-number version of a.c. circuits cannot be undone, note that the impedance Z in your textbook is actually $|\mathbf{Z}|$ here. From now on, you can forget the formula $Z = \sqrt{R^2 + X^2}$ because it's just the modulus of the complex impedance $\mathbf{Z}$, i.e. $|\mathbf{Z}|$. Also, forget $\tan\phi = \frac{\omega L}{R}$ because ϕ is actually the argument of $\mathbf{Z}$, i.e. $\arg(\mathbf{Z})$.

7.2. Lead or Lag?

Complex impedance also offer a neat alternative to the question 'Why inductors have voltage lead current and capacitors have voltage lagging current?' After this section, you won't have them switched around.

With the complex version of Ohms law

$$\mathbf{V} = \mathbf{IZ} \tag{42}$$

$$\text{Ph}(\mathbf{V}) = \text{Ph}(\mathbf{I}) + \text{Ph}(\mathbf{Z}) \tag{43}$$

For RC circuit, $\text{Ph}(\mathbf{Z}) = -90°$, so voltage lags current. For RL circuit, $\mathbf{Z} = \hat{j}(\omega L)$, $\text{Ph}(\mathbf{Z}) = +90°$, so voltage leads current. For RLC circuits, compute the $\text{Ph}(\mathbf{Z})$ and see if it's greater or smaller than $0°$.

7.3. Complex Power

You've seen P=VI in real form, the complex number version of power is $\mathbf{S}$, where

$$\mathbf{S} = \mathbf{V}_{rms}\overline{\mathbf{I}_{rms}} \tag{44}$$

Due to equation (42), the above equation can be rewritten as

$$\mathbf{S} = (\mathbf{I}_{rms}\mathbf{Z})\overline{\mathbf{I}_{rms}} = |\mathbf{I}_{rms}|^2\mathbf{Z} \tag{45}$$

Inductors and capacitors contributes only to imaginary component of any complex quantity in your calculations. So, real power is $\text{Re}(\mathbf{S})$, and reactive power is $\text{Im}(\mathbf{S})$.

7.3.1. Resonance

As we know ideal capacitors and inductors doesn't dissipate energy[17], maximum energy transfer occur when all the power is dissipated through resistor R. To make it happen, we want[18]. the reactive power, $\text{Im}(\mathbf{S})=0$.

On the right hand side of equation (45), only $\mathbf{Z}$ is complex. So, to maximize power dissipated (by resistor R) in a simple series RLC circuit,

$$\text{Im}(\mathbf{Z}) = \omega L - \frac{1}{\omega C} = 0 \tag{46}$$

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[17]they just store energy

[18]From inequality (24), max $\text{Re}(\mathbf{S}) = |\mathbf{S}|$. Illustrated by the right angle triangle, the only way to achieve this is to make the imaginary component zero so all the magnitude is dedicated to real component.

After solving, we get the condition for resonance:

$$\omega = \frac{1}{\sqrt{LC}} \tag{47}$$

7.3.2. Average Power

The equation $P_{avg} = V_{rms}I_{rms}\cos\theta$ in your textbook for average power dissipated (Tip: 'dissipated' always mean resistors only) might lead you to think of dot products. It isn't surprising at all because complex numbers and vectors are related. I don't recommend secondary school students to think of it as dot product because it involves geometric thinking of vector space or ideas in abstract linear algebra.

Writing equation (44), in polar form,

$$\mathbf{S} = |\mathbf{V}_{rms}|e^{\hat{j}\arg(\mathbf{V}_{rms})}\overline{|\mathbf{I}_{rms}|e^{\hat{j}\arg(\mathbf{I}_{rms})}} \tag{48}$$

Simplified,

$$\mathbf{S} = |\mathbf{V}_{rms}||\mathbf{I}_{rms}|e^{\hat{j}[\arg(\mathbf{V}_{rms})-\arg(\mathbf{I}_{rms})]} \tag{49}$$

Now we want the real part of $\mathbf{S}$,

$$P_{avg} = \text{Re}(\mathbf{S}) = |\mathbf{V}_{rms}||\mathbf{I}_{rms}|Re\left\{e^{\hat{j}[\arg(\mathbf{V}_{rms})-\arg(\mathbf{I}_{rms})]}\right\} \tag{50}$$

To save space, we write the phase difference between voltage and current as ϕ, where $\phi = \arg(\mathbf{V}_{rms}) - \arg(\mathbf{I}_{rms})$.

With Euler's formula (14),

$$P_{avg} = \text{Re}(\mathbf{S}) = |\mathbf{V}_{rms}||\mathbf{I}_{rms}|Re(\cos\phi + \hat{j}\sin\phi) \tag{51}$$

Simplifying,

$$P_{avg} = \text{Re}(\mathbf{S}) = |\mathbf{V}_{rms}||\mathbf{I}_{rms}|\cos\phi \tag{52}$$

Equation (49) and (50) can be skipped if you think in terms of De Moivre's theorem.

As you can see, just by mastering complex numbers, there are no new concept to assimilate here and no new formulae to remember.

I recommend Sharp EL-506 series calculators because it can compute complex numbers and perform many functions not available on classical Casio models.

8. FUN STUFF

Give a pair of integer values that satisfy: $a = b^a - (a-b)^{a-b}$