QUICK REVIEW OF REAL ANALYSIS

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1. REAL LINE / EUCLIDIAN SPACE

Thm (Archimedian property):

$$\exists n \in \mathbb{Z}^+ : nx > y, \quad x \in \mathbb{R}^+$$

Rmk: This is the same as saying real line is complete.
Trick: $\exists m : m - 1 \leq nx \leq m$

Thm (Cauchy-Schwarz): $\sum |\cdot|^2 \geq |\sum \cdot|^2$
Thm (Triangle Inequality): on $\int$ too!

1.1. Tricks

$$\epsilon - \delta : \frac{\delta}{2^n} \quad \text{for G.S.}$$
$$1 + nx_n \leq (1 + x_n)^n$$
$$\binom{n}{k} p^k < (1 + x_n)^n$$
$$b^n - a^n < (b - a)nb^{n-1}$$

2. TOPOLOGY

Thm: $\cup^\infty$ countable = countable
Ex. (Cantor's diagonal process) Let $\{s_n\}$be any sequences of 1 and 0, can always find s with its n^{th} digit being the flip of s_n, hence such sequences are <u>uncountable</u>.

Def (Limit point): every neighbor of it contains at least a point in the set. **Thm**: there are infinitey many. **Why**? finitely many points means can find a neighbor without it.

Def (Dense): every point in the set is a limit pt.

Def (Perfect): closed + all elements are its limit points[1]

Def (Open):
1) open $\Leftrightarrow$ $\overline{\text{open}}$=closed
2) $\forall x$ in the set, $\exists \delta > 0 : (x - \delta, x + \delta)$ is a subset

[1] i.e. no isolated points

Property: infinite $\cap$ <u>open</u> set and <u>$\cup$ closed</u> sets might[2] become otherwise. **Trick/Proof**: $[-\frac{1}{n} + Y, Y + \frac{1}{n})$

Def (Compact): every open cover contains <u>finite</u> subcover
Cor: infinite $\cap$(non-empty compact <u>$\searrow$nest</u>) = non-empty
Thm: compact <u>$\searrow$nest</u> with measure $\rightarrow 0$ has infinite $\cap$ being a singleton

Thm (Heine-Borel) compact $\Leftrightarrow$ closed + bounded

3. CANTOR SET

Properties: compact, perfect, uncountable, measure zero

4. SEQUENCES

Thm (MCT): monotone + bounded[3] $\leftrightarrow$ converges

Thm (Bolzano-Weistrass): every bounded sequence has a converging subsequence

Thm: <u>sub</u>sequential limits of sequence in the space form a closed subset of it

Thm: <u>compact</u> space/set $\Leftrightarrow$ for each sequence in it, $\exists$ a subsequence converging to a point in it

Def (Cauchy Sequences)

$$\forall \epsilon > 0, \exists N \in \mathbb{Z} : d(p_n, p_m) < \epsilon \quad \text{when } n, m \geq N$$

Def (Complete): every Cauchy sequence converges
Thm (Cauchy): convergent sequence $\Leftrightarrow$ Cauchy sequence
Rmk: completeness axiom $\rightleftharpoons$ Bolzano-Weistrass

4.1. Examples

1. $\{s_n\} = \mathbb{Q}$, then every point in $\mathbb{R}$ is a subseq limit,

$$\limsup_{n\to\infty} s_n = \infty, \quad \liminf_{n\to\infty} s_n = -\infty$$

[2] use $1/n$ edge argument
[3] in the same direction

2.

$$s_n = \frac{(-1)^n}{1+\frac{1}{n}}: \qquad \limsup_{n\to\infty} s_n = 1, \quad \liminf_{n\to\infty} s_n = -1$$

3. Commonly used limits:

$$\lim_{n\to\infty} \frac{1}{n^p} = 0, \qquad p \in \mathbb{R}^+$$
$$\lim_{n\to\infty} \sqrt[n]{p} = 1, \qquad p \in \mathbb{R}^+$$
$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$
$$\lim_{n\to\infty} \frac{n^a}{(1+p)^n} = 0, \qquad p \in \mathbb{R}^+, a \in \mathbb{R}$$

5. SERIES

Thm: convergent series $\Leftrightarrow$ tail approaches 0
Thm: $\sum$ converges $\Rightarrow$ sequence approaches 0

Thm: series of terms ≥ 0 converge $\Leftrightarrow$ partial sums form bounded sequence

Thm: absolute convergence $\Rightarrow$ converges. **Why?** Triangle Inequality. **Rmk:** may conditionally converges like $\sum \frac{(-1)^n}{n}$

Thm: without absolute convergence, infinite series can be rearranged to converge to anything.

5.1. Examples

$$\sum 2^k a_{2^k} \text{ converges for } a_k \downarrow \text{monotonically}$$
$$\sum \frac{1}{n^p} \text{ converges iff } p > 1$$
$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \text{ converges iff } p > 1$$
$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x = \lim_{n\to\infty} (1+\frac{x}{n})^n$$

Alternating haromic series converges conditionally

$\sum \frac{1}{n^p}$ and $\sum \frac{1}{n(\log n)^p}$: plug in $n = 2^k$, use $\sum 2^k a_{2^k}$

6. CONTINUITY

Def (Continuity):
1) $f(x \to a) = f(a)$;
2) $\forall \epsilon > 0, \exists \delta > 0 : d(x,a) < \delta \Rightarrow d(f(x), f(a)) < \epsilon$
3) limits from both sides match
4) lsc+usc

Notation: f continuous on E denoted by $f \in \mathscr{C}(E)$

Thm: continuity transverses as long as the functions are continuous on intermediate sets.

Thm[4]: a function is continuous on the set iff its inverse preserves open/closed-ness. i.e.,

$$f \in \mathscr{C}(\text{open/closed}) \Leftrightarrow f^{-1}(\text{open/closed}) = \text{open/closed}$$

Thm: $+ - \times \div$ preserves continuity across functions
Thm: continuous mapping preserves compactness

Thm: continuous function on compact metric space, it has a max and a min.

Thm (IVT): $f \in \mathscr{C}(I) \Rightarrow f(I)$ is interval too
Thm: connectedness preserved thro' continuous mapping
Thm[5]: $f \in \mathscr{C}(X) \Rightarrow \exists p, q \in X : f(q) \leq f(x) \leq f(p)$
i.e. must have values in between un-leveled points

Def (Uniform continuity): cannot have δ depend on ϵ or a
Rmk: The domain often determines the 'uniform' part!

Thm: For compact set, countinuity $\to$ uniform continuity

Thm: function preserving Cauchy sequence is continuous
Thm: Cauchy sequence preserved + bounded domain $\Leftrightarrow$ uniform continuity.

Thm: Monotone function can only have countable or finite[6] discontinuities

6.1. Examples

1. Continuous at 0, discontinuous (no one-side limits) everywhere else:
$$f(x) = x\delta(x - \mathbb{Q})$$

2. Continuous everywhere, except discontinuous at 0 (no one-side limits):

$$f(x) = (1 - \delta(x)) \sin \frac{1}{x}$$

6.2. Tricks

$$\sup_{t\in(a,x)} f(t) = f(x-), \qquad x \in (a,b)$$
$$\inf_{t\in(x,b)} f(t) = f(x+), \qquad x \in (a,b)$$

[4] useful characterization of continuity or open/closed sets!
[5] open version by replacing with strict inequality
[6] not trivial, see Rudin p.97

7. FUNCTIONAL CONVERGENCE

Def: $f_n \to f$, f is continuous if

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$

Def (Pointwise convergence): $f_n(x) \to f(x), \forall x \in E$
Def (Uniform convergence): u.cv.

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : |f_{n\geq N}(x) - f(x)| \leq \epsilon, \quad \forall x$$

Thm (Uniformly Cauchy): alternative u.cv. definition[7]

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : |f_{n\geq N}(x) - f_{m\geq N}(x)| \leq \epsilon, \quad \forall x$$

Thm (Weistrass M-Test): Let $|f_n(x)| \leq M_n$,

M_n converges $\Rightarrow \sum f_n$ converges uniformly

Thm: parameter and point limits exchangable,

$$\lim_{x\to x_o}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to x_o} f_n(x)$$

Thm: uniform convergence preserves continuity

Thm: Let f_n, f be continuous on compact[8] set,

pointwise + ($f_n \searrow_n$ nest) $\Rightarrow$ uniform convergence

Thm: limit is exchangable with integration/differentiation under uniform convergence

Def (Pointwise bounded): $|f_n(x)| < h(x) < \infty$
Def (Uniformly bounded): $|f_n(x)| < M$

Thm (Bolzano-Weistrass): holds for functional convergence[9] on countable set with pointwise, *non-uniform* boundedness.
Cor: uniform convergence of bounded functions implies uniform boundedness

Def (Equicontinuity): for a family $\mathscr{F}$ of functions f:

$$\forall \epsilon, \exists \delta : d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon, \forall f \in \mathscr{F}$$

Rmk: the *single* δ must work $\forall f \in \mathscr{F}$
Thm (Arzela-Ascoli): for compact metric space, $\mathscr{F}$ is compact $\Leftrightarrow \mathscr{F}$ is closed + uniformly bounded + equicontinuous.

7.1. Examples

1. Convergent series of continuous functions can have discontinuous sum:

$$f_n(x) = \frac{x^2}{(1+x^2)^n}, \qquad n \in \mathbb{N}$$
$$f(x) = \sum_{n=0}^{\infty} f_n(x) = (1-\delta(x))(1+x^2)$$

2. Everywhere discontinuous limit function, hence not Riemann-integrable:

$$f_m(x) = \lim_{n\to\infty}(\cos m!\pi x)^{2n}$$
$$\lim_{m\to\infty}\lim_{n\to\infty}(\cos m!\pi x)^{2n} = \delta(x - \mathbb{Q})$$

This is because $\forall m, f_m(x \in \mathbb{Q}^c) = 0$; and if $x \in \mathbb{Q}$, $m!x \in \mathbb{N}$ if $m \geq q$

3. Inexchangable limit and derivative, derivative sequence diverges as $n \to \infty$:

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}$$

4. Inexchangable limit and integral, $\int_0^1$ diverges [left], or reach different value [right]:

$$f_n(x) = n^2 x(1-x^2)^n \quad \text{or} \quad nx(1-x^2)^n$$

5. Uniformly bounded function that fail to have a converging functional subsequence: $f_n(x) = \sin nx$

Shown by contradiction, *if* $\exists$ subsequence n_k

$$\lim_{k\to\infty} |\underbrace{\sin n_k x - \sin n_{k+1} x}_{\to 0 \text{ by definition}}|^2 = 0$$

By LDCT[10], $\int \lim = \lim \int = 0$, but

$$\int_0^{2\pi} |\sin n_k x - \sin n_{k+1} x|^2 = 2\pi$$

6. Convergent functional sequence without uniformly convergent functional subsequence:

$$f_n(x) = \frac{x^2}{x^2 + (1-nx)^2}, \in [0,1] \quad \forall x \in [0,1]$$

7. Dirac-delta formulation[11] is not u.cv. on [0,1].
Rmk: Mass escaping to ∞ in whatever axis are not u.cv.

[7] Proof: $|a-b| \leq |a-c| + |c-b|$
[8] Counter-example: $1/(nx+1)$
[9] might not be uniform
[10] see measure theory
[11] triangular or rectangular. Infinitely delayed unit-step too.