Frequency Domain Interpretation of COLA

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Abstract

The derivation of Poisson summation formula and the alternative view of COLA criterion uses IDTFT, which are basically time domain equations. Therefore, I find it more like algebra than frequency domain intuition. To my belief that every LTI system has a frequency domain interpretation, I spent a whole day deriving as I repeatedly end up with the same throttle. Luckily Harvey and Ed cleared it for me the night before final exam.

1 Spectral Interpretation of COLA

The original time domain COLA formula [COLA(R)] is:

$$\sum_m w(n - mR) = 1 \tag{1}$$

The frequency domain version of COLA(R) is Nyquist($\frac{2\pi}{R}$), which translates mathematically as:

$$\sum_m \frac{1}{R} W(m\frac{2\pi}{R})\delta(\omega - m\frac{2\pi}{R}) = \delta(\omega) \tag{2}$$

The left hand side of the equation has only described shifts $\delta(\omega - m\frac{2\pi}{R})$. By matching terms with the right hand side of the equation, $W(m\frac{2\pi}{R})$ must be zero for $m \neq 0$. Mind that $m \in \mathbb{Z}$ so that you won't accidentally extend the results here to strong COLA.

Also, note that the only non-zero term in the summation on the left hand side of the equation is when $m = 0$. So, the non-zero term is $\frac{1}{R}W(0)$, and it equals to 1 by the equation above, hence:

$$\frac{1}{R}W(0) = 1$$

For COLA(R), $1 = \sum_m w(n - mR)$, so you get

$$\sum_m w(n - mR) = \frac{1}{R}W(0)$$

1.1 Derivation

This could be done by transforming both sides of the (1) into the frequency domain. First,

$$w(n) * \sum_m \delta(n - mR) = 1 \tag{3}$$

We know $w(n) \leftrightarrow W(\omega)$, and convolution transforms to multiplication. What about $\sum_m \delta(n - mR)$? Impulse train transforms to impulse train, except the frequency spacing and the amplitude are scaled. Here it is:

$$\sum_m \delta(n - mR) \leftrightarrow \frac{2\pi}{R} \sum_m \delta(\omega - m\frac{2\pi}{R}) \tag{4}$$

This transform pair is common in sampling theory, but the derivation is not easy. However, they could be found in most beefy signal processing books.

With all these Fourier pairs, we take the DTFT of (3):

$$W(\omega) \sum_m \frac{2\pi}{R} \delta(\omega - m\frac{2\pi}{R}) = 2\pi\delta(\omega) \tag{5}$$

After cancelling 2π on both sides, equation (5) can be integrated (sifted) into (2)

$$\sum_m \frac{1}{R} W(m\frac{2\pi}{R})\delta(\omega - m\frac{2\pi}{R}) = \delta(\omega)$$

2 Acknowledgement

I'd like to thank Edgar Berdahl at CCRMA, Stanford University for pointing out that the complex exponentials sitting in the opposite side of the unit circle cancels out. (I removed the derivation here because it is difficult to explain the $\frac{2\pi}{R}$ amplitude scaling by infinite summation. However, it gives an alternative view why the impulses are spaced by $\frac{2\pi}{R}$ in ω.)

Also, I'd like to thank Harvey Thornbu (now an Assistant Professor at Arizona State University) for reminding me of the impulse train to impulse train transformation.

Without their help, I'll never close the gap for the proof.